Formule di Teoria dei Segnali

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Formule di trigonometria

$$
\begin{aligned}
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha \\
\cos^2\alpha &= \frac{1+\cos 2\alpha}{2} \\
\sin^2\alpha &= \frac{1-\cos 2\alpha}{2} \\
\sin 2\alpha &= 2\sin\alpha\cos\alpha \\
\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha \\
\cos\alpha\cos\beta &= \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \\
\sin\alpha\sin\beta &= \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)] \\
\sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]
\end{aligned}
$$

Formule di Eulero

$$\cos\alpha = \frac{e^{j\alpha}+e^{-j\alpha}}{2} \qquad \sin\alpha = \frac{e^{j\alpha}-e^{-j\alpha}}{2j} \qquad e^{j\alpha} = \cos\alpha + j\sin\alpha$$

Proprietà $\delta(t)$ e $\delta(n)$

$\int_{t_1}^{t_2} x(t)\,\delta(t)\,dt = \begin{cases} x(0) & 0 \in (t_1,t_2) \\ 0 & \text{altrimenti} \end{cases}$	$\delta(n) = \begin{cases} 1 & n=0 \\ 0 & \text{altrimenti} \end{cases}$
$\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$	$\sum_{k=-\infty}^{+\infty} \delta(n-k) = 1$
$\int_{-\infty}^{+\infty} x(t)\delta(t-t_0)\,dt = x(t_0)$	$\sum_{n=-\infty}^{+\infty} x(n)\delta(n-n_0) = x(n_0)$
$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$	$x(n)\,\delta(n-n_0) = x(n_0)\,\delta(n-n_0)$
$\delta(t) = \delta(-t)$	$\delta(n) = \delta(-n)$
$\int_{-\infty}^{+\infty} x(\alpha)\delta(t-\alpha)\,d\alpha = x(t) * \delta(t) = x(t)$	$\sum_{k=-\infty}^{+\infty} x(k)\delta(n-k) = x(n) * \delta(n) = x(n)$
$\int_{-\infty}^{t} \delta(\tau)\,d\tau = \mathrm{u}(t) \quad \leftrightarrow \quad \delta(t) = \frac{d\mathrm{u}(t)}{dt}$	$\sum_{k=-\infty}^{n} \delta(k) = \mathrm{u}(n) \quad \leftrightarrow \quad \delta(n) = \mathrm{u}(n) - \mathrm{u}(n-1)$

Formule di utilità

$$\sum_{n=0}^{+\infty} \alpha^n = \frac{1}{1-\alpha} \qquad |\alpha| < 1 \qquad\qquad \sum_{n=M}^{N} \alpha^n = \begin{cases} \frac{\alpha^M - \alpha^{N+1}}{1-\alpha} & \alpha \neq 1 \\ N - M + 1 & \alpha = 1 \end{cases}$$

Media temporale per segnali aperiodici (1) e per segnali periodici (2)

$$\begin{array}{llll} (1) & < x(t) > = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, dt & & < x(n) > = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n) \\ (2) & < x(t) > = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, dt & & < x(n) > = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x(n) \end{array}$$

a) *Invarianza temporale* $y(t) = x(t - t_0) \implies < y(t) >=< x(t) >$

$y(n) = x(n - n_0) \implies < y(n) >=< x(n) >$

b) *Linearità* $z(\cdot) = a\, x(\cdot) + b\, y(\cdot) \implies < z(\cdot) >= a < x(\cdot) > +b < y(\cdot) >$

Potenza per segnali aperiodici (1) e per segnali periodici (2) ed Energia (3)

$$\begin{array}{llll} (1) & P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt & & P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \\ (2) & P_x = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2\, dt & & P_x = \frac{1}{N_0} \sum_{n=0}^{N_0-1} |x(n)|^2 \\ (3) & E_x = \int_{-\infty}^{+\infty} |x(t)|^2\, dt & & E_x = \sum_{n=-\infty}^{+\infty} |x(n)|^2 \end{array}$$

Potenza ed Energia mutua

$$\begin{array}{ll} P_{xy} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, y^*(t)\, dt & P_{xy} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)\, y^*(n) \\ E_{xy} = \int_{-\infty}^{+\infty} x(t)\, y^*(t)\, dt & E_{xy} = \sum_{n=-\infty}^{+\infty} x(n)\, y^*(n) \end{array}$$

a) *Invarianza temporale* $y(t) = x(t - t_0) \implies P_y = P_x$ e $E_y = E_x$

$y(n) = x(n - n_0) \implies P_y = P_x$ e $E_y = E_x$

b) *Non Linearità* $z(\cdot) = x(\cdot) + y(\cdot) \implies P_z = P_x + P_y + 2\,\mathrm{Re}[P_{xy}]$

$\implies E_z = E_x + E_y + 2\,\mathrm{Re}[E_{xy}]$

<u>**Funzione di autocorrelazione**</u> **per segnali di potenza aperiodici (1) e periodici (2) e per segnali di energia (3)**

(1) $$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^*(t-\tau)\, dt \qquad R_x(m) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)\, x^*(n-m)$$

(2) $$R_x(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, x^*(t-\tau)\, dt \qquad R_x(m) = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x(n)\, x^*(n-m)$$

(3) $$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)\, x^*(t-\tau)\, dt \qquad R_x(m) = \sum_{n=-\infty}^{+\infty} x(n)\, x^*(n-m)$$

<u>**Funzione di mutua correlazione**</u> **per segnali di potenza (1) e per segnali di energia (2)**

(1) $$R_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, y^*(t-\tau)\, dt \qquad R_{xy}(m) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)\, y^*(n-m)$$

(2) $$R_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t)\, y^*(t-\tau)\, dt \qquad R_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, y^*(n-m)$$

a) *Valore nell'origine* $R_x(0) = \begin{cases} E_x \\ P_x \end{cases}$ $\qquad R_{xy}(0) = \begin{cases} E_{xy} \\ P_{xy} \end{cases}$

b) *Simmetria coniugata* $R_x(\cdot) = R_x^*(-(\cdot))$ $\qquad R_{xy}(\cdot) = R_{yx}^*(-(\cdot))$

c) *Limitatezza* $|R_x(\cdot)| \le R_x(0)$ $\qquad |R_{xy}(\cdot)| \le \begin{cases} \sqrt{E_x E_y} \\ \sqrt{P_x P_y} \end{cases}$

<u>**Sistemi LTI nel dominio del tempo**</u>

$$y(t) = \int_{-\infty}^{+\infty} x(\alpha)\, h(t-\alpha)\, d\alpha = x(t) * h(t) \qquad y(n) = \sum_{k=-\infty}^{+\infty} x(k)\, h(n-k) = x(n) * h(n)$$

a) *Proprietà commutativa* $x(\cdot) * h(\cdot) = h(\cdot) * x(\cdot)$

b) *Proprietà distributiva* $x(\cdot) * [h_1(\cdot) + h_2(\cdot)] = x(\cdot) * h_1(\cdot) + x(\cdot) * h_2(\cdot)$

c) *Proprietà associativa* $x(\cdot) * [h_1(\cdot) * h_2(\cdot)] = [x(\cdot) * h_1(\cdot)] * h_2(\cdot)$

d) *Proprietà associativa mista* $a[x(\cdot) * h(\cdot)] = [ax(\cdot)] * h(\cdot) = x(\cdot) * [ah(\cdot)]$

e) *Invarianza temporale* $x(t - t_1) * h(t - t_2) = y(t - (t_1 + t_2))$

$x(n - n_1) * h(n - n_2) = y(n - (n_1 + n_2))$

Sistema non dispersivo $\iff$ $h(\cdot) = k\delta(\cdot)$

Sistema causale $\iff$ $h(t) = 0$ per $t < 0$ $\qquad h(n) = 0$ per $n < 0$

Sistema stabile $\iff$ $\int_{-\infty}^{+\infty} |h(t)|\, dt < \infty$ $\qquad \sum_{n=-\infty}^{+\infty} |h(n)| < \infty$

Serie di Fourier

Sintesi $\quad x(t) = \sum_{k=-\infty}^{+\infty} X_k\, e^{j2\pi k f_0 t} \qquad x(n) = \sum_{k=0}^{N_0-1} X_k\, e^{j2\pi k \nu_0 n}$

Analisi $\quad X_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, e^{-j2\pi k f_0 t} dt \qquad X_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x(n)\, e^{-j2\pi k \nu_0 n}$

$$x(\cdot) \text{ reale} \longrightarrow X_{-k} = X_k^* \longleftrightarrow \begin{cases} |X_{-k}| = |X_k| \\ \angle X_{-k} = -\angle X_k \end{cases}$$

1) *Linearità* $z(\cdot) = ax(\cdot) + by(\cdot) \longleftrightarrow Z_k = aX_k + bY_k$

2) *Traslazione temporale* $y(t) = x(t - t_0) \longleftrightarrow Y_k = X_k\, e^{-j2\pi k f_0 t_0}$

$y(n) = x(n - n_0) \longleftrightarrow Y_k = X_k\, e^{-j2\pi k \nu_0 n_0}$

3) *Riflessione* $y(\cdot) = x(-\cdot) \longleftrightarrow Y_k = X_{-k}$

4) *Derivazione* $y(t) = \frac{dx(t)}{dt} \longleftrightarrow Y_k = j2\pi k f_0 X_k$

5) *Differenza prima* $y(n) = x(n) - x(n-1) \longleftrightarrow Y_k = (1 - e^{-j2\pi k \nu_0}) X_k$

6) *Relazione di Parseval*

$\frac{1}{T_0} \int_{T_0} |x(t)|^2\, dt = \sum_{k=-\infty}^{+\infty} |X_k|^2 \qquad \frac{1}{N_0} \sum_{<N_0>} |x(n)|^2 = \sum_{k=<N_0>} |X_k|^2$

Trasformata di Fourier

$$\textit{Sintesi} \quad x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{j2\pi ft}\, df \qquad\qquad x(n) = \int_{-1/2}^{1/2} X(\nu)\, e^{j2\pi\nu n}\, d\nu$$

$$\textit{Analisi} \quad X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi ft}\, dt \qquad\qquad X(\nu) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j2\pi\nu n}$$

$$x(\cdot)\ \text{reale} \quad \longrightarrow \quad X(-(\cdot)) = X^*(\cdot) \quad \longleftrightarrow \quad \begin{cases} |X(-(\cdot))| = |X(\cdot)| \\ \angle X(-(\cdot)) = -\angle X(\cdot) \end{cases}$$

1) *Linearità*	$a_1x_1(\cdot) + a_2x_2(\cdot)$	$\longleftrightarrow$	$a_1X_1(\cdot) + a_2X_2(\cdot)$
2) *Riflessione*	$x(-(\cdot))$	$\longleftrightarrow$	$X(-(\cdot))$
3) *Cambiamento di scala*	$x(at)$	$\longleftrightarrow$	$\frac{1}{\lvert a\rvert} X\left(\frac{f}{a}\right)$
4) *Espansione*	$x\left[\frac{n}{M}\right]$	$\longleftrightarrow$	$X(M\nu)$
5) *Decimazione*	$x(Mn)$	$\longleftrightarrow$	$\frac{1}{M}\sum_{k=0}^{M-1} X\left(\frac{\nu-k}{M}\right)$
6) *Convoluzione*	$x(\cdot) * y(\cdot)$	$\longleftrightarrow$	$X(\cdot)Y(\cdot)$
7) *Prodotto*	$x(t)y(t)$	$\longleftrightarrow$	$X(f) * Y(f)$
	$x(n)y(n)$	$\longleftrightarrow$	$X(\nu) * Y(\nu) = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(u)Y(\nu - u)\, du$
8) *Derivazione d.d.t*	$\frac{d^k x(t)}{dt^k}$	$\longleftrightarrow$	$(j2\pi f)^k X(f)$
9) *Differenza prima*	$x(n) - x(n-1)$	$\longleftrightarrow$	$(1 - e^{-j2\pi\nu})X(\nu)$
10) *Derivazione d.d.f*	$t^k x(t)$	$\longleftrightarrow$	$\left(\frac{j}{2\pi}\right)^k \frac{d^k X(f)}{df^k}$
11) *Integrazione*	$\int_{-\infty}^{t} x(\alpha)\, d\alpha$	$\longleftrightarrow$	$\frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f)$
12) *Somma corrente*	$\sum_{k=-\infty}^{n} x(k)$	$\longleftrightarrow$	$\frac{X(\nu)}{1-e^{-j2\pi\nu}} + \frac{1}{2}X(0)\widetilde{\delta}(\nu)$
13) *Traslazione d.d.t*	$x(t - t_0)$	$\longleftrightarrow$	$X(f)\, e^{-j2\pi f t_0}$
	$x(n - n_0)$	$\longleftrightarrow$	$X(\nu)\, e^{-j2\pi\nu n_0}$
14) *Traslazione d.d.f*	$x(t)\, e^{j2\pi f_0 t}$	$\longleftrightarrow$	$X(f - f_0)$
	$x(n)\, e^{j2\pi\nu_0 n}$	$\longleftrightarrow$	$X(\nu - \nu_0)$

15) *Modulazione*	$x(t)\cos(2\pi f_0 t+\theta)$	$\longleftrightarrow$	$\frac{1}{2}X(f-f_0)e^{j\theta}+\frac{1}{2}X(f+f_0)e^{-j\theta}$
	$x(t)\cos(2\pi\nu_0 n+\theta)$	$\longleftrightarrow$	$\frac{1}{2}X(\nu-\nu_0)e^{j\theta}+\frac{1}{2}X(\nu+\nu_0)e^{-j\theta}$
16) *Campionamento d.d.f*	$\sum_{n=-\infty}^{+\infty} x(t-nT)$	$\longleftrightarrow$	$\sum_{k=-\infty}^{+\infty} \frac{1}{T}X\left(\frac{k}{T}\right)\delta\left(f-\frac{k}{T}\right)$
	$\sum_{k=-\infty}^{+\infty} x(n-kn)$	$\longleftrightarrow$	$\sum_{k=0}^{N-1} \frac{1}{N}X\left(\frac{k}{N}\right)\widetilde{\delta}\left(f-\frac{k}{N}\right)$
17) *Campionamento d.d.t*	$\sum_{n=-\infty}^{+\infty} x(nT)\delta(t-nT)$	$\longleftrightarrow$	$\sum_{k=-\infty}^{+\infty} \frac{1}{T}X\left(f-\frac{k}{T}\right)$
	$\sum_{k=-\infty}^{+\infty} x(kN)\delta(n-kN)$	$\longleftrightarrow$	$\sum_{k=0}^{N-1} \frac{1}{N}X\left(f-\frac{k}{N}\right)$
18) *Valore nell'origine*	$X(0)=\int_{-\infty}^{+\infty} x(t)\,dt$		$x(0)=\int_{-\infty}^{+\infty} X(f)\,df$
	$X(0)=\sum_{n=-\infty}^{+\infty} x(n)$		$x(0)=\int_{-1/2}^{+1/2} X(\nu)\,d\nu$

19) *Relazione di Parseval*

$\int_{-\infty}^{+\infty} |x(t)|^2\,dt=\int_{-\infty}^{+\infty} |X(f)|^2\,df$ $\qquad$ $\sum_{n=-\infty}^{+\infty} |x(n)|^2=\int_{-1/2}^{+1/2} |X(\nu)|^2\,d\nu$

<u>Trasformate notevoli</u> (segnali tempo continuo)

1) *Impulso rettangolare*	$A\,\mathrm{rect}\left(\frac{t}{T}\right)$	$\longleftrightarrow$	$AT\,\mathrm{sinc}(fT)$
2) *Impulso triangolare*	$A\,\Lambda\left(\frac{t}{T}\right)$	$\longleftrightarrow$	$AT\,\mathrm{sinc}^2(fT)$
3) *Esponenziale monolatero*	$A\,e^{-t/T}\mathrm{u}(t)$	$\longleftrightarrow$	$\frac{AT}{1+j2\pi fT}$
	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha/T}\mathrm{u}(t)$	$\longleftrightarrow$	$\frac{1}{(\alpha+j2\pi f)^n}$
4) *Esponenziale bilatero*	$A\,e^{-\lvert t\rvert/T}$	$\longleftrightarrow$	$\frac{2T}{1+(2\pi fT)^2}$
5) *Funzione sinc*	$A\,\mathrm{sinc}(2Bt)$	$\longleftrightarrow$	$\frac{A}{2B}\,\mathrm{rect}\left(\frac{f}{2B}\right)$
6) *Impulso ideale*	$\delta(t)$	$\longleftrightarrow$	1
7) *Segnale costante*	A	$\longleftrightarrow$	$A\,\delta(f)$
8) *Gradino unitario*	$\mathrm{u}(t)$	$\longleftrightarrow$	$\frac{1}{j2\pi f}+\frac{1}{2}\delta(f)$
9) *Funzione signum*	$\mathrm{sign}(t)$	$\longleftrightarrow$	$\frac{1}{j\pi f}$
10) *Fasore*	$A\,e^{j2\pi f_0 t}$	$\longleftrightarrow$	$A\,\delta(f-f_0)$

11) *Segnale coseno*	$A\cos(2\pi f_0 t)$	$\longleftrightarrow$	$\frac{A}{2}\,\delta(f-f_0)+\frac{A}{2}\,\delta(f+f_0)$
12) *Segnale seno*	$A\sin(2\pi f_0 t)$	$\longleftrightarrow$	$\frac{A}{2j}\,\delta(f-f_0)-\frac{A}{2j}\,\delta(f+f_0)$
13) *Treno di impulsi*	$\sum_{n=-\infty}^{+\infty}\delta(t-nT)$	$\longleftrightarrow$	$\frac{1}{T}\sum_{k=-\infty}^{+\infty}\delta\left(f-\frac{k}{T}\right)$

Trasformate notevoli (segnali tempo discreto)

1) *Impulso rettangolare*	$A\,\mathcal{R}_N(n)$	$\longleftrightarrow$	$\frac{\sin(\pi\nu N)}{\sin(\pi\nu)}e^{-j(N-1)\pi\nu}$
2) *Impulso triangolare*	$\mathcal{B}_{2N}(n)$	$\longleftrightarrow$	$\frac{\sin^2(\pi\nu N)}{N\,\sin^2(\pi\nu)}e^{-j2\pi N\nu}$
3) *Esponenziale monolatero*	$a^n\mathrm{u}(n)$	$\longleftrightarrow$	$\frac{1}{1-ae^{-j2\pi\nu}}$
4) *Esponenziale bilatero*	$a^{\|n\|}$	$\longleftrightarrow$	$\frac{1-a^2}{1-2a\cos(2\pi\nu)+a^2}$
5) *Funzione sinc*	$2\nu_c\,\mathrm{sinc}(2\nu_c n)$	$\longleftrightarrow$	$\mathrm{rep}_1\left[\mathrm{rect}\left(\frac{\nu}{2\nu_c}\right)\right]$
6) *Funzione sinc*^{2}	$2\nu_c\,\mathrm{sinc}^2(2\nu_c n)$	$\longleftrightarrow$	$\mathrm{rep}_1\left[\Lambda\left(\frac{\nu}{2\nu_c}\right)\right]$
7) *Impulso ideale*	$\delta(n)$	$\longleftrightarrow$	1
8) *Segnale costante*	A	$\longleftrightarrow$	$A\,\widetilde{\delta}(\nu)$
8) *Gradino unitario*	$\mathrm{u}(n)$	$\longleftrightarrow$	$\frac{1}{1-e^{-j2\pi\nu}}+\frac{1}{2}\widetilde{\delta}(\nu)$
9) *Funzione signum*	$\mathrm{sign}(n)$	$\longleftrightarrow$	$\frac{2}{1-e^{-j2\pi\nu}}$
10) *Fasore*	$A\,e^{j2\pi\nu_0 n}$	$\longleftrightarrow$	$A\,\widetilde{\delta}(\nu-\nu_0)$
11) *Segnale coseno*	$A\cos(2\pi\nu_0 n)$	$\longleftrightarrow$	$\frac{A}{2}\,\widetilde{\delta}(\nu-\nu_0)+\frac{A}{2}\,\widetilde{\delta}(\nu+\nu_0)$
12) *Segnale seno*	$A\sin(2\pi\nu_0 n)$	$\longleftrightarrow$	$\frac{A}{2j}\,\widetilde{\delta}(\nu-\nu_0)-\frac{A}{2j}\,\widetilde{\delta}(\nu+\nu_0)$
13) *Treno di impulsi*	$\sum_{k=-\infty}^{+\infty}\delta(n-kN)$	$\longleftrightarrow$	$\frac{1}{N}\sum_{k=-\infty}^{+\infty}\delta\left(\nu-\frac{k}{N}\right)$

Densità spettrale per segnali di potenza aperiodici (1) e periodici (2) e segnali di energia (3)

(1) $S_x(f)=\lim_{T\to\infty}\frac{1}{T}|X_T(f)|^2$ (2) $S_x(f)=\sum_{k=-\infty}^{+\infty}|X_k|^2\delta\left(f-\frac{k}{T_0}\right)$ (3) $S_x(f)=|X(f)|^2$